Finite acceptance of infinite words

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Abstract

In this paper we consider the following two types of finite acceptance of infinite words by finite automata: An infinite word ξ is accepted if and only if there is a run on input ξ for which an accepting state is visited at least once or at least once but only finitely often.
The resulting classes of regular ω-languages are characterized by language-theoretic means, and they are positioned into the known hierarchies of regular ω-languages.

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Languages of infinite words (so-called ω-languages) were introduced by Büchi [Bü62] and Trakhtenbrot [Tr62] in order to study decidability properties of monadic second order arithmetic. Since then several different types of accepting conditions for finite automata have been considered [Mu63], [MN66], [La69], [TB70], [SW74] and [TY83] (see also the surveys [St87], [Th90] and [EH93]). Recently, Yamasaki and Moriya ([MY88] and [Ya89]) investigated the classes of ω-languages accepted by finite automata when an infinite word (ω-word) is accepted iff an accepting state is visited at most finitely many times (including never). In particular, it was shown there that for completely specified, deterministic automata this class coincides with the class of complements of ω-languages accepted by deterministic Büchi automata having a single accepting state.

In this paper we consider mainly the following kind of finite acceptance of infinite words by finite automata: an infinite word ξ is accepted if and only if there is a run on input ξ for which an accepting state is visited only finitely often but at least once. This corresponds in a way to the unfair behaviour of processes where an action continuously enabled from some moment on is carried out only finitely many times (cf. [Fr86]).

It turns out that this type of acceptance essentially differs from all previously known ones, that is, except for the case of incompletely specified nondeterministic automata, defines new classes of accepted ω-languages. Moreover, in contrast to the classes defined by the basic types of acceptance introduced in [La69] and [SW74], none of these new classes are topologically characterizable.

The investigation of ω-languages accepted in the above described way is closely related to the investigation of ω-languages accepted by infinite runs containing at least one accepting state. Therefore, we study both types of acceptance in parallel, thereby recalling some previously known results (cf. [La69], [TB70], [SW74], [St76], [Wa76] and [TY83]).

The paper is organized as follows. In the next section we present some necessary notation and a short introduction into basic concepts of our paper.

Then Section 2 starts with the definition of finite acceptance. Here we also introduce some fundamental relationships between the classes of ω-languages accepted by the various modes of finite acceptance, in particular, we prove a projection lemma linking deterministically and nondeterministically accepted ω-languages.

Subsequently, in Section 3 we deal with language-theoretical characterizations of the previously defined classes of ω-languages. We obtain several inclusion relations between them and a relation to the classes investigated by Yamasaki and Moriya ([MY88] and [Ya89]).

The aim of the fourth section is to show that all the presented inclusion relations are proper and further ones do not hold. We prove here also that some of the classes do not possess natural closure properties with respect to union and intersection.

1 Notation and basic properties of ω-languages

In this section we introduce the notation used throughout the paper. Further we give some basic results from the theory of ω-languages which are necessary for our investigations. Additional information on the theory of ω-languages can be obtained from the quoted above papers.

By $\mathbb{N} = \{0, 1, 2, \ldots\}$ we denote the set of natural numbers. We consider the space X^ω of infinite strings (sequences) on a finite alphabet of cardinality card$X \leq 2$. We may consider a $\xi \in X^\omega$ also as a mapping $\xi : \{1, 2, \ldots\} \mapsto X$, thus $\xi(n)$ is the n^{th} letter ($n > 0$) of the sequence $\xi \in X^\omega$. By X^* we denote the set (monoid) of finite strings (words) on X, including the *empty* word λ. For $w \in X^*$ and $b \in X^* \cup X^\omega$ let $w \cdot b$ be their *concatenation.* This concatenation product extends in an obvious way to subsets $W \subseteq X^*$ and $B \subseteq X^* \cup X^\omega$. As usual we denote subsets of X^* as languages and subsets of X^ω as ω-languages. For a language W let $W^0 := \{\lambda\}$ and $W^{i+1} := W^i \cdot W$. Then $W^* := \bigcup_{i \in \mathbb{N}} W^i$ is the *submonoid* of X^* generated by W, and by W^ω we denote the set of infinite strings formed by concatenating in W. Furthermore $|w|$ is the *length* of the word $w \in X^*$, hence $X^n = \{w : w \in X^* \ \wedge \ |w| = n\}$.

$\mathsf{A}(B) := \{w : w \in X^* \ \wedge \ \exists(b(b \in X^* \cup X^\omega w \cdot b \in B)\}$ is the set of all *initial words (prefixes)* of the set $B \subseteq X^* \cup X^\omega$. For the sake of brevity we shall write $w \cdot B, W \cdot b$ and $\mathsf{A}(b)$ instead of $\{w\} \cdot B, W \cdot \{b\}$ and $\mathsf{A}(\{b\})$ respectively, and we shall abbreviate the fact that w is an initial word of b, that is $w \in \mathsf{A}(b)$, by $w \sqsubseteq b$. Moreover, we call $B \subseteq X^* \cup X^\omega$ *prefix-free* iff $w \sqsubseteq b$ and $w, b \in B$ imply $w = b$.

We consider X^ω as a topological space with the basis $(w \cdot X^\omega)_{w \in X^*}$. Since X is finite, this topological space is homeomorphic to the Cantor discontinuum, hence compact. Open sets in X^ω are of the form $W \cdot X^\omega$ where $W \subseteq X^*$. From this follows that a subset $F \in X^\omega$ is *closed* iff $\mathsf{A}(\beta) \subseteq \mathsf{A}(F)$ implies $\beta \in F$.

The topological closure of subset $F \subseteq X^\omega$, that is, the smallest closed subset of (X^ω, ρ) containing F is denoted by $\mathcal{C}(F)$. It holds $\mathcal{C}(F) = \{\xi : A(\xi) \subseteq A(F)\}$.

Having defined open and closed sets in X^ω, we proceed to the next classes of the Borel hierarchy (cf. [Ku66]):

> $\mathbf{F}_\sigma$ is the set of countable unions of closed subsets of X^ω, and
>
> $\mathbf{G}_\delta$ is the set of countable intersections of open subsets of X^ω.

To be consistent with this notation, we denote the classes of open (closed) subsets of X^ω by $\mathbf{G}$ and $\mathbf{F}$, respectively.

If X and Y are finite alphabets, we consider the product alphabet $X \times Y$ and the projections from $X \times Y$ into X and Y, defined by $pr((x, y)) := x$, and $pr_2((x, y)) := y$, respectively. The extensions of pr and pr_2 to $(X \times Y)^* \cup (X \times Y)^\omega$ are also denoted by

pr and pr_2, respectively. As it is well known [Ku66], projections on $(X \times Y)^\omega$ preserve open, closed and, hence, $\mathbf{F}_\sigma$ -sets.

We mention here that Borel classes are also closed under (finite) union and intersection.

An automaton is a quadruple $\mathcal{A} = (X, Q, I, \delta)$ where Q is a (finite) set of states, I is a nonempty subset of Q (the initial states), δ is a mapping of $Q \times X$ into $\mathcal{P}(Q) := \{Q' : Q' \subseteq Q\}$ (the transition relation).

$\mathcal{A}$ is said to be *totally defined* if $\delta(q, x) \neq \emptyset$ for each pair $(q, x) \in Q \times X$, and $\mathcal{A}$ is said to be *deterministic* if $I = \{q_0\}$ and there is at most one state in $\delta(q, x)$. Whenever $\mathcal{A}$ is deterministic, the single state contained in $\delta(q, a)$ is also denoted by $f(q, x)$, and we shall also write $\mathcal{A} = (X, Q, q_0, f)$ instead of $\mathcal{A} = (X, Q, I, \delta)$.

We extend δ in the usual way to a mapping $\delta : Q \times X^* \to \mathcal{P}(Q)$:

$$\delta(q, \lambda) := \{q\} \text{ and } \delta(q, w \cdot x) := \bigcup_{q' \in \delta(q,w)} \delta(q', x) \text{ for } x \in X.$$

Let $\mathcal{A} = (X, Q, I, \delta)$ be an automaton and let $D \subseteq Q$. We say that the pair $(\mathcal{A}, D)$ accepts a language $W \subseteq X^*$ iff $W = \{w : \exists q_0(q_0 \in I \ \wedge \ \delta(q_0, w) \cap D \neq \emptyset\}$ (short: $W = T(\mathcal{A}, D)$), and we say that a language $W \subseteq X^*$ is *regular* provided $W = T(\mathcal{A}, D)$ for some finite automaton $\mathcal{A}$ and some subset D of Q. It is well-known that for every regular language there is a totally defined deterministic automaton such that $W = T(\mathcal{A}, D)$ for some set D.

Let ξ be an ω-word, a *complete run* of $\mathcal{A}$ on ξ is a sequence $\zeta \in (X \times Q)^\omega$ of pairs $(\xi(i), q_i) \in X \times Q$ such that $\exists q_0(q_0 \in I \ \wedge \ \forall i(i > 0 \to q_i \in \delta(q_{i-1}, \xi(i))))$. When $\zeta \in (X \times Q)^\omega$ is a complete run of $\mathcal{A}$ on ξ the sequence $pr_2\zeta$ is called a *run* of $\mathcal{A}$ on ξ. The set of all runs of $\mathcal{A}$ on ξ is denoted by $\mathsf{run}_\mathcal{A}(\xi)$, or simply by $\mathsf{run}(\xi)$, if there is no danger of confusion.

We say that a (finite) automaton $\mathcal{A} = (X, Q, I, \delta)$ accepts (under $D \subseteq Q$) an ω-language $F \subseteq X^\omega$ iff

$$F = \{\xi : \exists q \exists r(q \in D \wedge r \in \mathsf{run}_\mathcal{A}(\xi) \wedge q \text{ occurs infinitely often in } r\} \ ,$$

and we call an ω-language $F \subseteq X^\omega$ *regular* if there are a finite automaton $\mathcal{A}$ and a set $D \subseteq Q$ such that $\mathcal{A}$ accepts F under D (short $F = T_\omega(\mathcal{A}, D)$).

For a finite alphabet X the class of all regular subsets of X^ω is denoted by $REG(X^\omega)$. We mention some properties of the class of regular ω-languages.

Property 1 [Bü62] *$REG(X^\omega)$ is closed under union, intersection complement and pre-multiplication by regular languages, and moreover, if $E \in REG((X \times Y)^\omega)$ then prE is also regular.*

Unlike the case of languages it is known that not all regular ω-languages are representable in the form $T_\omega(\mathcal{A}, D)$ for a deterministic automaton $\mathcal{A}$. We have only the following relationship to a topological property.

Property 2 [La69] *A regular ω-language $E \subseteq X^\omega$ is accepted by a deterministic automaton $\mathcal{A} = (X, Q, q_0, f)$ iff $E \in \mathbf{G}_\delta$.*

We add some further relations to the classes of the Borel hierarchy.

Property 3 [La69] *Every regular ω-language $L \subseteq X^\omega$ is in the Boolean closure of the Borel class $\mathbf{G}_\delta$.*

Property 4 *If $L \subseteq X^\omega$ is regular then $\mathsf{A}(L)$ is a regular language.*

¿From Property 4 a characterization of closed regular ω-languages follows immediately.

Corollary 5 *For a ω-language $F \subseteq X^\omega$ the following properties are equivalent:*

1. *F is regular and closed.*
2. *There is a (partially defined) finite automaton $\mathcal{A} = (X, Q, I, \delta)$ such that $F = \{\xi : \mathsf{run}_\mathcal{A}(\xi) \neq \emptyset\}$, that is, $F = T_\omega(\mathcal{A}, Q)$.*
3. *There is a (partially defined) deterministic finite automaton $\mathcal{A}(X, Q, q_0, f)$ such that $F = \{\xi : \mathsf{run}_\mathcal{A}(\xi) \neq \emptyset\}$, that is, $F = T_\omega(\mathcal{A}, Q)$.*

2 Finite acceptance of ω-languages

In this section we introduce modes of accepting infinite words based on only finitely many occurrences of final states in the corresponding runs of the automata. We give some fundamental relations between the classes of ω-languages accepted by finite automata under various acceptance modes.

Let $r \in Q^\omega$ be a run of $\mathcal{A} = (X, Q, q_0, \delta)$ on an ω-word ξ. Then we set

$Ex(r) := \{q \in Q : q = r(n) \text{ for some } n\}$, and

$Fin(r) := \{q \in Q : q = r(n) \text{ for at least one but only finitely many } n\}$.

For a subset D of Q we define the following two modes of ω-languages finitely accepted by the pair $(\mathcal{A}, D)$:

$B_e(\mathcal{A}, D) := \{\xi : \exists r(r \in \mathsf{run}(\xi) \ \wedge \ Ex(r) \cap D \neq \emptyset)\}$, and

$B_f(\mathcal{A}, D) := \{\xi : \exists r(r \in \mathsf{run}(\xi) \ \wedge \ Fin(r) \cap D \neq \emptyset)\}$.

Now, for each of the two above defined acceptance modes B_e and B_f, according to the types of finite automata involved, we define four classes of ω-languages.:

$TDB_y := \{B_y(\mathcal{A}, D) : \mathcal{A} = (X, Q, q_0, f)$ is a totally defined deterministic finite automaton and $D \subseteq Q\}$,

$DB_y := \{B_y(\mathcal{A}, D) : \mathcal{A} = (X, Q, q_0, f)$ is a deterministic finite automaton and $D \subseteq Q\}$,

$TB_y := \{B_y(\mathcal{A}, D) : \mathcal{A} = (X, Q, I, \delta)$ is a totally defined finite automaton and $D \subseteq Q\}$,

$B_y := \{B_y(\mathcal{A}, D) : \mathcal{A} = (X, Q, I, \delta)$ is a finite automaton and $D \subseteq Q\}$.

We remark here that the classes of ω-languages TDB_y and TB_y, where y is one of the letters e or f, depend on which alphabet X we choose, e.g. $\{a,b\}^\omega \in TDB_e$ if $X = \{a,b\}$, but, as we shall see later, $\{a,b\}^\omega \notin TB_e$ if $X = \{a,b,c\}$.

The B_e-classes defined here were investigated before in [TB70], [St76], [Wa76], but as they have close connections to the B_f-classes it is useful to derive their properties in parallel.

Since for a totally defined finite automaton $\mathcal{A} = (X, Q, I, \delta)$ the identity

$$B_e(\mathcal{A}, D) = T(\mathcal{A}, D) \cdot X^\omega \tag{1}$$

holds, the following characterization of the classes TB_e and TDB_e is immediate.

Property 6

$$\begin{aligned}\{E : E \subseteq X^\omega \wedge E \in TDB_e\} &= \{E : E \subseteq X^\omega \wedge E \in TB_e\} \\ &= \{W \cdot X^\omega : W \subseteq X^* \wedge W \ \textit{is regular}\} \\ &= \{E : E \subseteq X^\omega \ \textit{and}\ E \ \textit{is regular and open}\}\end{aligned}$$

Next we derive examples of ω-languages in the classes DB_e and TDB_f showing that these classes do not coincide with $TDB_e = TB_e$.

Example 1 Consider the partially defined automaton $\mathcal{A}_1 = (\{a,b\}, \{q_0, q_1\}, q_0, f)$ where

$$\begin{aligned} f(q_0, a) &= q_0, \text{ and} \\ f(q_0, b) &= f(q_1, a) = q_1 \ . \end{aligned}$$

Then $B_e(\mathcal{A}_1, \{q_1\}) = aa^*ba^\omega$ which is obviously not in TDB_e. □

Example 2 For the fully defined automaton $\mathcal{A}_2 = (\{a,b\}, \{q_0, q_1, q_2\}, q_0, f)$ where

$$\begin{aligned} f(q_0,a) &= f(q_0,b) = q_1 \ , \\ f(q_1,a) &= f(q_2,a) = q_2, \text{ and} \\ f(q_1,b) &= f(q_2,b) = q_1 \end{aligned}$$

we have $B_f(\mathcal{A}_2, \{q_1\}) = \{a,b\}^* \cdot a^\omega$. This ω-language is not in $\mathbf{G}_\delta$, in particular $\{a,b\}^* \cdot a^\omega = T_\omega(\mathcal{A}, D)$ forces $\mathcal{A}$ to be nondeterministic. □

We mention the following canonical inclusion relations between B_f-classes and B_e-classes.

Proposition 7 *$TDB_e \subseteq TDB_f$, $TB_e \subseteq TB_f$, $DB_e \subseteq DB_f$, and $B_e \subseteq B_f$.*

Proof. Let $\mathcal{A} = (X, Q, q_0, \delta)$ be an automaton and let $D \subseteq Q$. Then we consider the automaton $\mathcal{A}' = (X, Q \times \{0,1\}, I \times \{0\}, \delta')$ and $D' = D \times \{0\}$, where δ' is defined by :

$$\delta'((q,b),x) := \begin{cases} (\delta(q,x), b), & \text{if } q \neq D, \quad \text{and} \\ (\delta(q,x), 1), & \text{if } q \in D, \quad \text{for } q \in Q, b \in \{0,1\} \text{ and } x \in X. \end{cases}$$

Clearly, if $\mathcal{A}$ is totally defined or deterministic then $\mathcal{A}'$ is also totally defined or deterministic, respectively.

Informally, $\mathcal{A}'$ works as follows: $\mathcal{A}'$ starts with the second coordinate of its states being 0 and simulates in its first coordinate the automaton $\mathcal{A}$. As long as $\mathcal{A}$ meets no final state $q \in D$ the automaton $\mathcal{A}'$ does not change the second coordinate, but once $\mathcal{A}$ meets a final state $q \in D$ the second coordinate is switched to 1 and never changed back to 0, that is, $\mathcal{A}'$ avoids to enter a further final state.

Thus every run of $\mathcal{A}'$ contains at most one final state, and there is exactly one final state iff the corresponding run of A contains at least one final state. Consequently, $B_f(\mathcal{A}', D') = B_e(\mathcal{A}, D)$.

Q.E.D.

We mention still the following easily verified closure properties of the "nondeterministic" classes TB_e, B_e, TB_f and B_f, respectively.

Property 8 *The classes TB_e, B_e, TB_f and B_f are closed under union and projection.*

It is well-known (see e.g. [EH93], [St84], [St93], [WS77]) that the ω-languages accepted by deterministic and nondeterministic automata in several classes can be related using so-called projection lemmas. Here we show that this is also true in the case of the B_e- and B_f-classes. Our proof follows the line of § 4 in [St76] where this result was shown for B_e-classes.

Lemma 9 (Projection Lemma) *Let y be one of the letters e, or f, and let $E \in B_y$ ($E \in TB_y$, respectively). Then there are an alphabet Q and an ω-language $L \subseteq (X \times Q)^\omega$ such that $L \in DB_y$ ($L \in TDB_y$, respectively) and $E = pr\ L$.*

Proof: First we consider the case of partially defined automata.
Let $\mathcal{A} = (X, Q, I, \delta)$ be a nondeterministic automaton and $D \subseteq Q$, such that $B_y(\mathcal{A}, D) = E$.

Define a deterministic automaton $\mathcal{D} := (X \times Q, Q \cup \{q_0\}, q_0, f)$, where $q_0 \notin Q$, via:

$$f(q_0, (x, q')) := \begin{cases} q' & \text{, if } \exists q(q \in I \ \wedge \ q' \in \delta(q, x)), \text{ and} \\ \text{not defined} & \text{, otherwise.} \end{cases}$$

and for $q \in Q$:

$$f(q, (x, q')) := \begin{cases} q' & \text{, if } q' \in \delta(q, x), \text{ and} \\ \text{not defined} & \text{, otherwise.} \end{cases}$$

Then, by construction, there is an infinite run r of $\mathcal{D}$ on $\zeta \in (X \times Q)$ if and only if ζ is a complete run of $\mathcal{A}$ on $pr\zeta$ and, moreover, $r = pr_2\ \zeta$.

This observation proves the identity $\mathsf{run}_{\mathcal{A}}(\xi) = \bigcup\{\mathsf{run}_{\mathcal{D}}(\zeta) : \zeta$ is a complete run of $\mathcal{A}$ on $\xi\}$.

Thus $\xi \in B_y(\mathcal{A}, D)$ iff there is a complete run ζ of $\mathcal{A}$ on ξ and $\zeta \in B_y(\mathcal{D}, D)$, what verifies our assertion for partially defined automata.

In the case of a totally defined nondeterministic automaton for every $x \in X$ and $q \in Q$ we fix states $s_{0,x} \in \bigcup_{q \in I} \delta(q, x)$ and $s_{q,x} \in \delta(q, x)$ and we define the deterministic automaton $\mathcal{D} := (X \times Q, Q \cup \{q_0\}, q_0, f)$, where $q_0 \notin Q$ via:

$$f(q_0, (x, q')) := \begin{cases} q' & \text{, if } \exists q(q \in I \ \wedge \ q' \in \delta(q, x)), \text{ and} \\ s_{0,x} & \text{, otherwise.} \end{cases}$$

and for $q \in Q$:

$$f(q, (x, q')) := \begin{cases} q' & \text{, if } q' \in \delta(q, x), \text{ and} \\ s_{q,x} & \text{, otherwise} \end{cases}$$

Repeating the above argument now yields only the inclusion $\mathsf{run}_{\mathcal{A}}(\xi) \subseteq \bigcup\{\mathsf{run}_{\mathcal{D}}(\zeta) : \zeta$ is a complete run of $\mathcal{A}$ on $\xi\}$. But since $s_{0,x} \in \delta(I, x)$ and $s_{q,x} \in \delta(q, x)$, every run r' of $\mathcal{D}$ on $\zeta \in (X \times Q)^\omega$ is also a run of $\mathcal{A}$ on $pr\zeta$, though r' does not necessarily coincide with $pr_2\zeta$, whence $\mathsf{run}_{\mathcal{A}}(\xi) \supseteq \bigcup\{\mathsf{run}_{\mathcal{D}}(\zeta) : \zeta$ is a complete run of $\mathcal{A}$ on $\xi\}$, and the assertion follows.

Q.E.D.

Next we give a relation between the classes of ω-languages accepted by totally defined and partial deterministic automata.

Lemma 10 *Let y be one of the letters e, or f. Then for every $L \in DB_y$ there are an $E \in TDB_y$ and a closed regular ω-language F such that*

$$L = E \cap F \ .$$

Conversely, if $E \in REG(X^\omega) \cap \mathbf{G}$ and $F \in REG(X^\omega) \cap \mathbf{F}$ then $L = E \cap F \in DB_e$.

Remark. As we shall see later in Example 3 for the class DB_f the converse does not hold.

Proof. Let $\mathcal{A} = (X, Q, q_0, f)$ be a partial deterministic automaton. We complete $\mathcal{A}$ by introducing a dead sink s whenever $f(q, x)$ is not defined, that is, we define $\mathcal{A}' := (X, Q \cup \{s\}, q_0, f')$ where $s \notin Q$, and

$$f'(q,x) := \begin{cases} f(q,x) & \text{, if } f(q,x) \text{is defined, and} \\ s & \text{, otherwise.} \end{cases}$$

In particular, $f'(s, x) = s$ for all $x \in X$. Thus if r is a run of $\mathcal{A}$ on ξ then r is also a run of $\mathcal{A}'$ on ξ, and $\mathcal{A}$ has no infinite run on ξ iff the run of $\mathcal{A}'$ on ξ has s^ω as suffix.

This shows, that $B_y(\mathcal{A}, D) = B_y(\mathcal{A}', D) \cap T_\omega(\mathcal{A}, Q)$. Now the assertion follows from Corollary 5.

In order to prove the converse for DB_e in view of Property 6 and Corollary 5 we may assume $E = B_e(\mathcal{A}, D)$ for some totally defined deterministic automaton $\mathcal{A} = (X, Q, q_0, f)$ and $F = T_\omega(\mathcal{B}, S)$ for some partially defined deterministic automaton $\mathcal{B} = (X, S, s_0, g)$. Then it is readily seen that $E \cap F = B_e(\mathcal{A} \times \mathcal{B}, D \times S)$ where $\mathcal{A} \times \mathcal{B} = (X, Q \times S, (q_0, s_0), f \times g)$ is the usual Cartesian product of the automata $\mathcal{A}$ and $\mathcal{B}$.

Q.E.D.

As an immediate consequence we obtain.

Corollary 11 *The class DB_e is closed under intersection.*

The ω-language $T_\omega(\mathcal{A}, Q)$ from the first part of the proof of Lemma 10 is a closed ω-language containing $B_y(\mathcal{A}, D)$. Thus $\mathsf{A}(B_y(\mathcal{A}, D)) \subseteq \mathsf{A}(T_\omega(\mathcal{A}, Q))$, and we obtain the following sufficient condition for ω-languages in DB_y to be already in TDB_y.

Corollary 12 *If $\mathsf{A}(L) = X^*$ and $L \in DB_y$ then $L \in TDB_y$.*

Proof. Since $\mathcal{A}$ is deterministic, $\mathsf{A}(T_\omega(\mathcal{A}, D)) = X^*$ implies that $\mathcal{A}$ is totally defined.

Q.E.D.

3 Language-theoretical characterizations

It is well-known (cf. [SW74], [LS77], [TY83], [St87]) that among the topologically defined subclasses of regular ω-languages the subclasses $REG(X^\omega) \cap \mathbf{G}$, $REG(X^\omega) \cap \mathbf{F}$, $REG(X^\omega) \cap \mathbf{F}_\sigma \cap \mathbf{G}_\delta$ and $REG(X^\omega) \cap \mathbf{F}_\sigma$ have characterizations via regular languages and operations like product and adherence, so-called language-theoretical characterizations. In this section we give similar characterizations of the classes DB_e, B_e, TDB_f, DB_f and B_f. It turns out that, except for B_f, the B_f-classes have no thorough characterization in terms of product and adherence.

First, we recall the just mentioned language-theoretical characterization of the topologically defined subclasses of $REG(X^\omega)$.

To this end we use operations transforming languages to ω-languages. The first one introduced in [SW74] and [LS77], usually called *adherence* of languages (cf. [BN80]), is defined as follows

$$\boldsymbol{ls} W := \{\xi : \xi \in X^\omega \wedge \mathsf{A}(\xi) \subseteq \mathsf{A}(W)\} \quad \text{for } W \subseteq X^* \ . \tag{2}$$

The second one is the usual premultiplication of X^ω by a language W, $W \cdot X^\omega$.

We briefly summarize the results in the following table.

Property 13

	Class	Representation	Comment
1.	$REG(X^\omega) \cap \mathbf{G}$	$W \cdot X^\omega$	*W regular*
2.	$REG(X^\omega) \cap \mathbf{F}$	$\boldsymbol{ls} W$	*W regular*
3.	$REG(X^\omega) \cap \mathbf{F}_\sigma \cap \mathbf{G}_\delta$	$\bigcup_{i=1}^{n} W_i \cdot \boldsymbol{ls} V_i$	*W_i, V_i regular and W_i prefix-free*
4.	$REG(X^\omega) \cap \mathbf{F}_\sigma$	$\bigcup_{i=1}^{n} W_i \cdot \boldsymbol{ls} V_i$	*W_i, V_i regular*

(Here e.g. the last line reads as follows: $F \in REG(X^\omega) \cap \mathbf{F}_\sigma$ if and only if there are an $n \in \mathbb{N}$ and regular languages W_i and V_i $(1 \leq i \leq n)$ such that $F = \bigcup_{i=1}^{n} W_i \cdot \boldsymbol{ls} V_i$.)

Before we proceed to a similar characterization of for ω-languages in DB_e we mention still the following properties of the operators A and $\boldsymbol{ls}$ (see [LS77],[BN80]):

The operations A and $\boldsymbol{ls}$ share some common properties: Let op be one of the operations A, or $\boldsymbol{ls}$, and let $U, V, W \subseteq X^*$, $B, B' \subseteq X^* \cup X^\omega$. Then

$$\begin{aligned} op(B \cup B') &= op(B) \cup op(B') \ , & (3)\\ op(W \cdot B) &= op(W) \cup W \cdot op(B) \quad \text{if } B \neq \emptyset, & (4)\\ op(\mathsf{A}(U) \setminus V \cdot X^*) &= op(U) \setminus V \cdot op(X^*), \text{ and} & (5) \end{aligned}$$

$$\boldsymbol{ls}\mathsf{A}(W) = \boldsymbol{ls}W = \boldsymbol{ls}\mathsf{A}(\boldsymbol{ls}W)\,, \tag{6}$$
$$\mathcal{C}(F) = \boldsymbol{ls}\mathsf{A}(F) \quad \text{if } F \subseteq X^\omega\,. \tag{7}$$

Proposition 14 [St76],[Wa76] *Let $L \subseteq X^\omega$. Then the following conditions are equivalent.*

1. *$L \in DB_e$*
2. *There are an $n \in \mathbb{N}$ and regular languages W_i, U_i $(1 \le i \le n)$ such that $\bigcup_{i=1}^{n} W_i$ is prefix-free and $L = \bigcup_{i=1}^{n} W_i \cdot \boldsymbol{ls}U_i$.*
3. *$L = E \cap F$ where $E \in REG(X^\omega) \cap \mathbf{G}$ and $F \in REG(X^\omega) \cap \mathbf{F}$.*
4. *L is regular and $L = E \cap F$ where E is open and F is closed.*
5. *L is regular and $\boldsymbol{ls}\mathsf{A}(L) \setminus L$ is a closed subset of X^ω.*

Proof. From Lemma 10 we know that 1. and 3. are equivalent, moreover, 3. trivially implies 4. Hence, it suffices to show the implications 1. $\Longrightarrow$ 2., 2. $\Longrightarrow$ 3., 4. $\Longrightarrow$ 5., and 5. $\Longrightarrow$ 3..

1. $\Longrightarrow$ 2. : Let $\mathcal{A} = (X, Q, q_0, f)$ be a deterministic automaton and D be a subset of Q. Then, clearly, we have $B_e(\mathcal{A}, D) = \bigcup_{q \in D} W_q \cdot \boldsymbol{ls}U_q$, where

$$\begin{aligned} W_q &:= \{w : w \in X^* \setminus \{\lambda\} \ \wedge\ f(q_0, w) = q \ \wedge\ \forall v(\lambda \sqsubset v \sqsubset w \to f(q_0, v) \notin D)\}, \text{ and} \\ U_q &:= \{v : v \in X^* \ \wedge\ f(q, v) \text{ is defined})\}\,. \end{aligned}$$

By definition $\bigcup_{q \in D} W_q$ is prefix-free, what proves our assertion.

2. $\Longrightarrow$ 3. : If $L = \emptyset$ the assertion is true. Therefore, we may assume $L = \bigcup_{i=1}^{n} W_i \cdot \boldsymbol{ls}U_i$, where U_i and W_i are nonempty regular languages with $\bigcup_{i=1}^{n} W_i$ prefix-free and $\boldsymbol{ls}U_i \neq \emptyset$ for all $i = 1, \ldots, n$.

Consider the smallest closed ω-language containing $L, \mathcal{C}(L) = \boldsymbol{ls}\mathsf{A}(L)$. From the above identities we obtain

$$\boldsymbol{ls}\mathsf{A}(L) = \boldsymbol{ls}\mathsf{A}(\bigcup_{i=1}^{n} W_i \cdot \boldsymbol{ls}U_i) = \boldsymbol{ls}\mathsf{A}(\bigcup_{i=1}^{n} W_i) \cup \bigcup_{i=1}^{n} W_i \cdot \boldsymbol{ls}U_i\ .$$

Since $\bigcup_{i=1}^{n} W_i$ is prefix-free, we have $\boldsymbol{ls}\mathsf{A}(\bigcup_{i=1}^{n} W_i) \cap \bigcup_{i=1}^{n} W_i \cdot X^\omega = \emptyset$, hence $L = \boldsymbol{ls}\mathsf{A}(L) \cap \bigcup_{i=1}^{n} W_i \cdot X^\omega$.

4. $\implies$ 5. : If $L = E \cap F$ with $E \in \mathbf{G}$ and $F \in \mathbf{F}$ then $F \supseteq \mathcal{C}(L) = \boldsymbol{ls}\mathsf{A}(L)$ because F is closed and contains L. Consequently, $L = E \cap \boldsymbol{ls}\mathsf{A}(L)$ and $\boldsymbol{ls}\mathsf{A}(L) \setminus L = \boldsymbol{ls}\mathsf{A}(L) \setminus E$ being the difference of a closed and an open subset of X^ω is itself closed.

5. $\implies$ 3. : In virtue of Properties 4 and 13.1 the ω-language $\boldsymbol{ls}\mathsf{A}(L)$ is regular provided L is regular. Thus $L = \boldsymbol{ls}\mathsf{A}(L) \setminus (\boldsymbol{ls}\mathsf{A}(L) \setminus L) = \boldsymbol{ls}\mathsf{A}(L) \cap (X^\omega \setminus (\boldsymbol{ls}\mathsf{A}(L) \setminus L))$ has the required form.

Q.E.D.

We can now estimate the projection closure of DB_e, B_e .

Proposition 15 $B_e = REG(X^\omega) \cap \mathbf{F}_\sigma$.

Proof. As $REG(X^\omega) \cap \mathbf{F}_\sigma$ is closed under projection and as $DB_e \subseteq REG(X^\omega) \cap \mathbf{F}_\sigma$ we have also $B_e \subseteq REG(X^\omega) \cap \mathbf{F}_\sigma$.

Let $F \in REG(X^\omega) \cap \mathbf{F}_\sigma$. Then according to Property 13.4 there are regular languages W_i, V_i such that $F = \bigcup_{i=1}^{n} W_i \cdot \boldsymbol{ls}V_i$. Since B_e is also closed under union and projection, it suffices to show that $W \cdot \boldsymbol{ls}V \in B_e$ for arbitrary regular languages $W, V \subseteq X^*$. For convenience we may assume that $\lambda \notin W$ or $W = \{\lambda\}$ (otherwise replace $W \cdot \boldsymbol{ls}V$ by $\boldsymbol{ls}V \cup (W \setminus \{\lambda\}) \cdot \boldsymbol{ls}V$).

If $W = \{\lambda\}$ we have $\boldsymbol{ls}V \in DB_e \subseteq B_e$. If $\lambda \notin W$ we consider the alphabet $Y := \{a, b\}$ and define languages $W' := W \times \{a\}^* \cdot b$ and $V' := V \times \{a\}^*$, where $W \times U$ is defined as the largest language $V \subseteq (X \times Y)^*$ such that $prV = W$ and $pr_2V = U$.

Clearly, W' and V' are regular and, moreover, W' is prefix-free, hence $W' \cdot \boldsymbol{ls}V' \in DB_e$.

By construction, $pr(W' \cdot \boldsymbol{ls}V') = W \cdot \boldsymbol{ls}V$, what finishes our proof.

Q.E.D.

In a similar way as in the direction 1. $\implies$ 2. of the proof of Proposition 14 we get a language-theoretical representation of the ω-languages in DB_f.

Proposition 16 *Let $E \in DB_f$. Then there are an $n \in \mathbb{N}$ and prefix-free and regular languages $W_i, V_i \subseteq X^* \setminus \{\lambda\}$ and regular languages $U_i \subseteq X^* (1 \le i \le n)$ such that*

$$E = \bigcup_{i=1}^{n} W_i \cdot V_i^* \cdot (\boldsymbol{ls}U_i \setminus V_i \cdot X^\omega) .$$

If, moreover, $E \in TDB_e$ then $U_i = X^*$ for all $i = 1, \ldots, n$.

Proof. Let $\mathcal{A} = (X, Q, q_0, f)$ be a deterministic automaton and D be a subset of Q. Then we have $B_f(\mathcal{A}, D) = \bigcup_{q \in D} W_q \cdot V_q^* \cdot (\boldsymbol{ls}U_q \setminus V_q \cdot X^\omega)$, where

$$W_q \ := \ \{w : w \in X^* \setminus \{\lambda\} \ \wedge \ f(q_0, w) = q \ \wedge \ \forall v (\lambda \sqsubset v \sqsubset w \to f(q_0, v) \neq q)\},$$

$$\begin{aligned} V_q &:= \{w : w \in X^* \setminus \{\lambda\} \ \wedge\ f(q,w) = q \ \wedge\ \forall v(\lambda \sqsubset v \sqsubset w \rightarrow f(q_0, v) \neq q)\}, \text{ and} \\ U_q &:= \{v : v \in X^* \ \wedge\ f(q,v) \text{ is defined})\} \end{aligned}$$

(When $\mathcal{A}$ is totally defined $U_q = X^*$.)

Q.E.D.

For a prefix-free language $V \subseteq X^* \setminus \{\lambda\}$ and an arbitrary $\xi \in X^\omega$ we have either $\xi \in V^\omega$ or there is an $n \in \mathbb{N}$ such that $\xi = v_1 \cdots v_n \cdot \xi'$ where $v_i \in V$ and $\xi' \notin V \cdot X^\omega$, that is,

$$V^* \cdot (X^\omega \setminus V \cdot X^\omega) = X^\omega \setminus V^\omega \ . \tag{8}$$

Thus we obtain the following.

Corollary 17 *Let $E \in TDB_f$. Then there are an $n \in \mathbb{N}$ and prefix-free and regular languages $W_i, V_i \subseteq X^* \setminus \{\lambda\}$, $(1 \le i \le n)$ such that*

$$E = \bigcup_{i=1}^{n} W_i \cdot (X^\omega \setminus V_i^\omega) \ .$$

As we shall see in the next section, the classes TDB_f and DB_f are not closed under union. From this fact and the subsequent Proposition 21 one infers that the converse to Proposition 16 is not valid, that is, there are ω-languages of the indicated form not belonging to the classes TDB_f or DB_f, respectively.

As an immediate consequence of Proposition 16 and the above identity $B_e = REG(X^\omega) \cap \mathbf{F}_\sigma$ (Proposition 15) we obtain the following characterization of the class B_f.

Corollary 18 $B_f = REG(X^\omega) \cap \mathbf{F}_\sigma$

Proof. Proposition 14 and the above proposition verify $DB_e \subseteq DB_f \subseteq REG(X^\omega) \cap \mathbf{F}_\sigma$. Thus $B_e \subseteq B_f \subseteq REG(X^\omega) \cap \mathbf{F}_\sigma$, and the assertion follows from the just mentioned identity $B_e = REG(X^\omega) \cap \mathbf{F}_\sigma$.

Q.E.D.

Since the product of two prefix-free regular languages is again a prefix-free and regular language, from the language-theoretic characterization of the classes TDB_e and DB_e (Proposition 14) one easily obtains that both classes are closed under premultiplication by prefix-free regular languages. Though the language-theoretic characterizations of the classes TDB_f and DB_f are not as satisfactory as the ones for TDB_e and DB_e, each of the mentioned classes is closed under premultiplication by prefix-free regular languages.

Proposition 19 *Let $W \subseteq X^*$ be a prefix-free and regular language, and let $E \subseteq X^\omega$ be an ω-language in DB_f (or TDB_f respectively). Then $W \cdot E \in DB_f$ ($\in TDB_f$ respectively).*

Proof. If $W \subseteq \{\lambda\}$ the assertion is obvious. Let $\mathcal{A} = (X, Q, q_0, f)$ be a deterministic automaton such that $E = B_f(\mathcal{A}, D)$ for some $D \subseteq Q$. Since $W \not\subseteq \{\lambda\}$ is prefix-free and regular, $W = T(\mathcal{B}, \{s_+\})$ for some totally defined finite deterministic automaton $\mathcal{B} = (X, S, s_0, g)$ and some $s_+ \in S$ where $s_0 \neq s_+$. Supposing $Q \cap S = \emptyset$ we define the composition $\mathcal{A} \uplus \mathcal{B}$ of $\mathcal{A}$ and $\mathcal{B}$ as follows:

$\mathcal{A} \uplus \mathcal{B} := (X, Q \cup S, s_0, h)$ where

$$h(z, x) := \begin{cases} g(z, x), & \text{if } z \in S \text{ and } z \neq s_+, \\ f(q_0, x), & \text{if } z = s_+, \text{and} \\ f(z, x), & \text{if } z \in Q. \end{cases}$$

We have that $h(z, x)$ is not defined if $z = s_+$ and $f(q_0, x)$ is not defined or if $z \in Q$ and $f(q_0, x)$ is not defined. Thus $\mathcal{A} \uplus \mathcal{B}$ is totally defined if $\mathcal{A}$ is totally defined.

Informally, the composition $\mathcal{A} \uplus \mathcal{B}$ starts as $\mathcal{B}$ and does not switch to $\mathcal{A}$ until $\mathcal{B}$ meets the final state s_+. After meeting s_+, it simulates the work of $\mathcal{A}$.

This informal description verifies immediately that $B_f(\mathcal{A} \otimes \mathcal{B}, D) = W \cdot E$.

Q.E.D.

Utilizing the Projection lemma we obtain immediately the following closure property of the class TB_f.

Corollary 20 *The class TB_f is closed under premultiplication with regular languages.*

Although the above Proposition 16 does not give a full characterization of the classes DB_f or TDB_f, the following proposition shows that a subclass of ω-languages satisfying this characterization belongs to DB_f or tdb_f, respectively

Proposition 21 *Let $W, V \subseteq X^* \setminus \{\lambda\}$ be prefix-free regular languages, and let $U \subseteq X^*$ be a regular language. Then*

$$W \cdot V^* \cdot (X^\omega \setminus V \cdot X^\omega) \in TDB_f \quad \textit{and} \quad W \cdot (V \cap \mathsf{A}(U))^* \cdot (\boldsymbol{ls}U \setminus V \cdot X^\omega) \in DB_f \ .$$

Proof. If $W = \emptyset$ or $U = \emptyset$ the assertion is trivially true, if $V = \emptyset$ we have in virtue of Eq. (1) $W \cdot X^\omega \in TDB_e \subseteq TDB_f$ and in virtue of Proposition 14 $W \cdot \boldsymbol{ls}U \in DB_e \subseteq DB_f$.

Assume now $W \neq \emptyset$, $V \neq \emptyset$ and $U \neq \emptyset$. Since $W, V \subseteq X^* \setminus \{\lambda\}$ are prefix-free and regular, $V = T(\mathcal{B}, \{s_+\})$ and $W = T(\mathcal{C}, \{z_+\})$) for some totally defined finite deterministic automata $\mathcal{B} = (X, S, s_0, g)$ and $\mathcal{C} = (X, Z, z_0, h)$ and states $s_+ \in S \setminus \{s_0\}$, $z_+ \in Z \setminus \{z_0\}$. Furthermore, we assume that $g(s, x) \neq s_0$ for all $s \in S$ and $x \in X$.

Without loss of generality we may assume $U = \mathsf{A}(U)$ and $\mathcal{A} = (X, Q, q_0, f)$ be a partially defined deterministic automaton such that $U = T(\mathcal{A}, Q)$. Then $\mathcal{A}$ is not totally defined unless $U = X^*$.

The following definition resembles the composition in the proof of Proposition 19:

$(\mathcal{A} \otimes \mathcal{B}) \uplus \mathcal{C} := (X, (Q \times S) \cup Z, z_0, f^{\otimes})$ where

$$\begin{aligned} f^{\otimes}(z,x) &:= \begin{cases} h(z,x), & \text{if } h(z,x) \neq z_+, \text{ and} \\ (q_0, s_0), & \text{if } h(z,x) = z_+, \end{cases} \quad , \text{ if } z \in Z \ , \\ f^{\otimes}((q,s),x) &:= \begin{cases} (q_0, s_0), & \text{if } g(s,x) = s_+, \text{ and } f(q,x) \text{ is defined,} \\ (f(q,x), g(s,x)), & \text{otherwise.} \end{cases} \end{aligned}$$

Thus $f^{\otimes}((q,s),x)$ is not defined if $f(q,x)$ is not defined, and $(\mathcal{A} \otimes \mathcal{B}) \uplus \mathcal{C}$ is totally defined if $\mathcal{A}$ is totally defined.

Informally, $(\mathcal{A} \otimes \mathcal{B}) \uplus \mathcal{C}$ starts as $\mathcal{C}$ and does not switch to $\mathcal{A} \otimes \mathcal{B}$ until C reaches its final state z_+. Then it simulates simultaneously the work of $\mathcal{A}$ and $\mathcal{B}$ with the only exception that both automata reinitialize themselves whenever $\mathcal{B}$ reaches its final state s_+.

We remark that due to our construction $f^{\otimes}(z_0, w) = (q_0, s_0)$ if and only if $w \in W \cdot (V \cap \mathsf{A}(U))^*$, and that $f^{\otimes}(z_0, v) \in Q \times S$ iff there is a $w \in W \cdot (V \cap \mathsf{A}(U))^*$ such that $v = w \cdot u$ and $u \in \mathsf{A}(U) \setminus V \cdot (X^* \setminus \{\lambda\})$. These observations show $B_f((\mathcal{A} \otimes \mathcal{B}) \uplus \mathcal{C}, \{(q_0, s_0)\}) = W \cdot (V \cap \mathsf{A}(U))^* \cdot (\boldsymbol{ls} U \setminus V \cdot X^{\omega})$.

Q.E.D.

In view of Eq. (8) the representation of Proposition for an ω-language $E \in TDB_f$ resembles in some sense the class

$$\mathbb{L}' = \{X^{\omega} \setminus W \cdot V^{\omega} : W, V \subseteq X^* \text{ and } W, V \text{ are prefix-free and regular}\}$$

investigated in [MY88] and [Ya89] [1]:

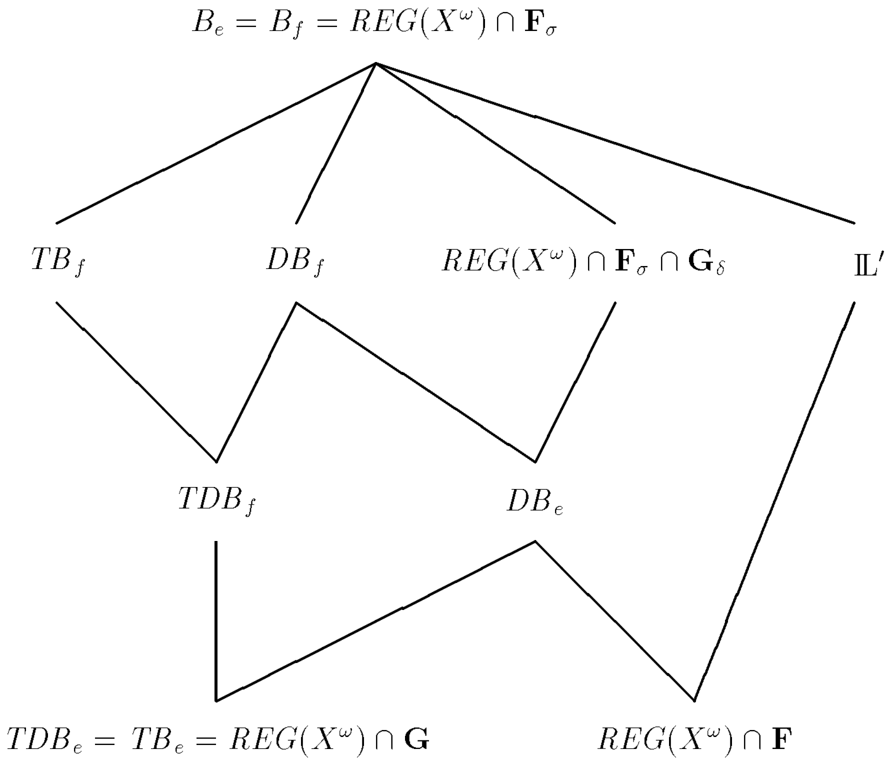
As it was shown in [MY88] this class satisfies the containment relations

$$\begin{aligned} REG(X^{\omega}) \cap \mathbf{F} &\subset \mathbb{L}' \subset REG(X^{\omega}) \cap \mathbf{F}_{\sigma} \text{ and} && (9) \\ REG(X^{\omega}) \cap \mathbf{G} &\not\subseteq \mathbb{L}' \ . && (10) \end{aligned}$$

This yields that, although the definition of $\mathbb{L}'$ given above resembles in some sense Proposition 21, the class $\mathbb{L}'$ is incomparable to TDB_f and TB_f, and since it contains the ω-language $\{a,b\}^{\omega} \setminus (a^* \cdot b)^{\omega} = \{a,b\} \cdot a^{\omega}$ it is also incomparable to DB_e and $REG(X^{\omega}) \cap \mathbf{F}_{\sigma} \cap \mathbf{G}_{\delta}$.

[1] The complementary class $\mathbb{L} := \{W \cdot V^{\omega} : W, V \subseteq X^* \text{ and } W, V \text{ are prefix-free and regular}\}$ investigated there is not contained in $REG(X^{\omega}) \cap \mathbf{F}_{\sigma}$.

So far we have proved identities and containment properties for families of B_e-classes, B_f-classes, Borel-subclasses of $REG(X^\omega) \cap \mathbf{F}_\sigma$ and the class $\mathbb{L}'$ depicted in the following diagram:



In order to verify that all inclusions presented are strict and that other inclusions do not hold we mention the following items.

1. In view of $X^* \cdot a^\omega \notin \mathbf{G}_\delta$ (e.g. [La69]) and $X^* \cdot a^\omega \in TDB_f$ (Example 2) we have $TDB_f \setminus REG(X^\omega) \cap \mathbf{F}_\sigma \cap \mathbf{G}_\delta \neq \emptyset$.
2. $REG(X^\omega) \cap \mathbf{F} \not\subseteq TB_f$ is shown in the below Corollary 26.
3. As we shall also see in Example 4 of the subsequent section the class DB_f (and hence also its subclass DB_e) does not contain all unions of the form $E \cup F$ where $E \in REG(X^\omega) \cap \mathbf{G}$ and $F \in (REG(X^\omega) \cap \mathbf{F}) \cup TDB_f$.

   Consequently, $TB_f \not\subseteq DB_f$, and $DB_e \subset REG(X^\omega) \cap \mathbf{F}_\sigma \cap \mathbf{G}_\delta \not\subseteq DB_f$.
4. Finally, below in Example 5 we show that the union $TB_f \cup DB_f$ is not closed under intersection with open regular ω-languages. This shows the proper inclusion $TB_f \cup DB_f \subset REG(X^\omega) \cap \mathbf{F}_\sigma$.

4 Further closure and nonclosure properties

In this section we shall derive mainly nonclosure properties of the classes TDB_f, DB_f and TB_f. The derived nonclosure properties verify that the type of finite acceptance we investigate defines new classes of regular ω-languages with, however, singular properties.

The following technical lemmas prove to be useful for the investigation of B_f-acceptance.

Lemma 22 *Let $\mathcal{A} = (X, Q, I, \delta)$ be a finite automaton, and let $D \subseteq Q$.*

1. *If $\xi \notin B_f(\mathcal{A}, D), r \in \mathsf{run}_{\mathcal{A}}(\xi)$ and if $r(|w|) \in D$ for some $w \sqsubset \xi$ then there is a $v \in X^* \setminus \{\lambda\}$ such that $w \cdot v \sqsubset \xi$ and $r(|w|) \in \delta(r(|w|), v)$.*
2. *If $w \cdot \eta \in B_f(\mathcal{A}, D), r \in \mathsf{run}(w \cdot \eta), Fin(r) \cap D \neq \emptyset, r(|w|) \in D$ and if $r(|w|) \in \delta(r(|w|), v)$ for some $v \in X^* \setminus \{\lambda\}$ then $w \cdot v^i \cdot \eta \in B_f(\mathcal{A}, D)$ for all $i \in \mathbb{N}$.*

Proof. The first assertion is just a reformulation of the fact that once an infinite run r of $\mathcal{A}$ on $\xi \notin B_f(\mathcal{A}, D)$ contains an accepting state $q \in D$ it has to contain this state q infinitely often, and the second one claims that an insertion of a finite loop into an infinite accepting run r of $\mathcal{A}$ does not change the accepting behaviour of $\mathcal{A}$.

Q.E.D.

Lemma 23 *Let $\mathcal{A} = (X, Q, q_0, f)$ be a deterministic automaton, $F = B_f(\mathcal{A}, D)$ and $w \in X^*$ such that $f(q_0, w) \in D$ and $\forall u(\lambda \sqsubset u \sqsubset w \rightarrow f(q_0, u) \notin D)$, and let $\eta \in X^\omega$ satisfy $w \cdot \eta \notin F$.*

If $f(q_0, w \cdot v) = f(q_0, w)$ for some $v \in X^* \setminus \{\lambda\}$ then $w \cdot v^i \cdot \eta \in F$ for all $i \in N$, but $w \cdot v^\omega \notin F$.

Proof. The assertion $w \cdot v^i \cdot \eta \in F$ is already mentioned in Lemma 22. The remaining one follows, because the run r of $\mathcal{A}$ on $w \cdot v^\omega$ consists of the initial part $(f(q_0, u))_{\lambda \sqsubset u \sqsubseteq w}$ not containing any final state plus an infinite repetition of the loop $(f(f(q_0, w), v'))_{v' \sqsubset v}$.

Q.E.D.

Our lemma has the following application.

Corollary 24 *Let $X = \{a, b\}, F \in DB_f$ such that $a \cdot a^* \cdot b^* \cdot a^\omega \subseteq F$ and $a^* \cdot b^\omega \setminus F$ is infinite. Then there are $i, j, k \in \mathbb{N}$ such that $j, k > 0$, $a^i \cdot (a^j \cdot b^k)^* \cdot a^\omega \subseteq F$ and $a^i \cdot (a^j \cdot b^k)^\omega \notin F$.*

Proof. Assume that $F = B_f(A, D)$ for some deterministic automaton $\mathcal{A} = (X, Q, q_0, f)$ and some $D \subseteq Q$. First observe that, since $a \cdot a^* \cdot b^* \cdot a^\omega \subseteq F$, $f(q_0, w)$ is defined for every $w \in a \cdot a^* \cdot b^* \cdot a^*$.

Since $a^\omega \in B_f(\mathcal{A}, D)$, there is a smallest $i \in \mathbb{N}, i \geq 1$ such that $f(q_0, a^i) = q \in D$. Since $a^* \cdot b^\omega \setminus B_f(A, D)$ is infinite, there is a $j > 0$ such that $a^{i+j} \cdot b^\omega \notin B_f(A, D)$. Now Lemma 22.1 implies $f(q, a^{i+j} \cdot b^k) = q$ for some $k > 0$, and the assertion follows from Lemma 23.

Q.E.D.

Thus the following ω-languages are not in DB_f :

Example 3 From Example 2 we know that $\{a, b\} \cdot a^\omega \in TDB_f$. The ω-language $a^* \cdot b^* \cdot a^\omega \cup a^* \cdot b^\omega$ is obviously regular and closed. Their intersection

$$F_3 := \{a, b\}^* \cdot a^\omega \cap (a^* \cdot b^* \cdot a^\omega \cup a^* \cdot b^\omega) = a^* \cdot b^* \cdot a^\omega$$

satisfies the assumption of Corollary 24, whence $F_3 \notin DB_f$.

This shows that, in contrast to the class DB_e, the class DB_f is not closed even under intersection with closed regular ω-languages. □

The next example considers the closure of DB_e under union.

Example 4 We consider the open regular ω-language $E := (a^*b)(b^* \cdot a) \cdot X^\omega$. Its union with $\{a, b\} \cdot a^\omega \in TDB_f$ is

$$F_4 := (a^*b)(b^* \cdot a) \cdot X^\omega \cup \{a, b\}^* \cdot a^\omega = (a^*b)(b^* \cdot a) \cdot X^\omega \cup a^\omega \ ,$$

which is also the union of E with the closed regular ω-language a^ω.

Again we observe that F_4 satisfies the assumption of Corollary 24, whence $F_4 \notin DB_f$.

Here $(a^*b)(b^* \cdot a) \cdot X^\omega \in REG(X^\omega) \cap \mathbf{G}$, $\{a, b\}^* \cdot a^\omega \in TDB_f$ and $a^\omega \in REG(X^\omega) \cap \mathbf{F}$.

A careful examination of this example shows that the classes DB_f and also DB_e are neither closed under union with open regular ω-languages nor under union with closed regular ω-languages. □

Next we show that that TB_f does not contain $REG(X^\omega) \cap \mathbf{F}$. To this end we investigate the relation of an ω-language $E \in TB_f$ to the smallest closed ω-language containing E, $\mathcal{C}(E) = \boldsymbol{ls}\mathsf{A}(E)$.

Lemma 25 *Let $E \subseteq X^\omega$ and $E \in TB_f$. Then for every $\xi \in E$ there is a prefix $w \sqsubset \xi$ such that $w \cdot X^\omega \subseteq \boldsymbol{ls}\mathsf{A}(E)$.*

Proof. Let $E = B_f(\mathcal{A}, D)$ for some finite automaton $\mathcal{A} = (X, Q, I, \delta)$ and some $D \subseteq Q$. If $\xi \in E$ there is an accepting run $r \in \mathsf{run}_{\mathcal{A}}(\xi)$. Let $w \sqsubset \xi$ such that $r(|w|) \in D$. If $w \cdot X^\omega \subseteq E$ we are done. Otherwise consider an arbitrary $\eta \in w \cdot X^\omega \setminus E$.

Since $\mathcal{A}$ is totally defined there is a run $r' \in \mathsf{run}_{\mathcal{A}}(\eta)$ such that $r(i) = r'(i)$ for $i = 1, \ldots, |w|$.

According to Lemma 22 there is a v_1 such that $w \cdot v_1 \sqsubset \eta$ and $r(|w|) \in \delta(r(|w|), v_1)$, and using the second part of Lemma 22, we obtain $w \cdot v_1 \cdot \xi' \in E$ when $w \cdot \xi' = \xi$.

Replacing successively w by $w \cdot v_1 \cdot \ldots \cdot v_i$ for $i = 1, 2, \ldots$ and repeating this argument, we get a family $(w \cdot v_1 \cdots v_i \cdot \xi')_{i=1,2\ldots} \subseteq E$ converging to η, that is, $\mathsf{A}(\eta) \subseteq \bigcup_{i=1,2\ldots} \mathsf{A}(w \cdot v_1 \cdots v_i \cdot \xi')$. Hence $\eta \in \boldsymbol{ls}\mathsf{A}(E)$.

Q.E.D.

Corollary 26 *If $F \in TB_f$ and F is closed then F is also open.*

In particular, our corollary verifies that for $\emptyset \neq Y \subset X$ the closed ω-language $Y\omega$ does not belong to TB_f.

Finally, we are going to show that neither the class TB_f nor its union with DB_f are closed under intersection. To this end we derive a third technical lemma.

Lemma 27 *Let $F \subseteq \{a, b, c\}^\omega$ be in $TB_f \cup DB_f$, and let $ba^*ba^\omega \cup bba^*c^*b^\omega \subseteq F$, $ba^\omega \notin F$ and $bba^*c^\omega \setminus F$ be an infinite set. Then*

$$\exists i, j (i, j \in \mathbb{N} \wedge j > 0 \wedge bba^i (c^j)^* a^\omega \subseteq F) \text{ or}$$

$$\exists i (i \in \mathbb{N} \wedge i > 0 \wedge b(a^i)^* ba^\omega \subseteq F) \ .$$

Proof. Consider an automaton $\mathcal{A} = (X, Q, I, \delta)$ which satisfies $F = B_f(\mathcal{A}, D)$ for some $D \subseteq Q$ and an accepting run r of $\mathcal{A}$ on $bba^\omega \in F$. Then there is a shortest prefix $w \sqsubset bba^\omega$ such that $q \in \delta(q_0, w) \cap D$ for some $q_0 \in I$. In the sequel we consider this finite prefix $\hat{r}$ of r of length $|w|$ ending with the state q.

First let $\mathcal{A}$ be totally defined.

If $w = b$ then, since $ba^\omega \notin F$ and $\mathcal{A}$ is totally defined, every run r' on ba^ω starting with $\hat{r} = q$ has to contain infinitely many occurrences of q, that is, there is at least one $i > 0$ such that $q \in \delta(q, a^i)$. Hence by Lemma 22 we have $b(a^i)^* ba^\omega \subseteq F$.

Similarly, if $w = bba^i$ then every run r'' on $bba^i c^\omega \notin F$ starting with $\hat{r}$ has infinitely many occurrences of q, that is, there is a $j > 0$ such that $q \in \delta(q, c^j)$. Again Lemma 22 shows that $bba^i (c^j)^* a^\omega \subseteq F$.

If $\mathcal{A}$ is not totally defined and deterministic then $\delta(q_0, w)$ is defined for every $w \in \mathsf{A}(F)$, in particular, for all $w \in ba^* \cup bba^*c^*$. Hence $\hat{r}$ continues to runs r' on ba^ω or r'' on $bba^i c^\omega$, respectively, and the above arguments are also valid.

Q.E.D.

Example 5 Consider the ω-languages $L_5 := b \cdot \{a,b,c\}^* \cdot b \cdot \{a,b\}^\omega$ and $E_5 := (\{bb\} \cup baa^*bb) \cdot \{a,b,c\}^\omega \in REG(X^\omega) \cap \mathbf{G}$.

The identity $L_5 = b \cdot (\{a,c\} \cup b \cdot \{a,b\}^* \cdot c)^* \cdot (\{a,b,c\}^\omega \setminus (\{a,c\} \cup b \cdot \{a,b\}^* \cdot c) \cdot \{a,b,c\}^\omega)$ verifies that according to Proposition 21 $L_5 \in TDB_f$.

Now it is readily seen $L_5 \cap E_5$ fulfills the assumption of Lemma 27, but we have $bba^i(c^j)^*a^\omega \cap L_5 = bba^\omega \not\supseteq bba^i(c^j)^*a^\omega$ and $b(a^i)^*ba^\omega \cap E_5 = bba^\omega \not\supseteq b(a^i)^*ba^\omega$. Consequently, $L_5 \cap E_5 \notin TB_f \cup DB_f$.

Thus the union of the classes TB_f and DB_f is not closed under intersection. As we have seen above, it holds even that $L \cap E \notin L \in TDB_f$ and $E \in REG(X^\omega) \cap \mathbf{G}$ do not necessarily imply $L \cap E \in TB_f \cup DB_f$, showing in particular that also TB_f cannot be closed under intersection.

$REG(X^\omega) \cap \mathbf{F}_\sigma$, however, is closed under intersection. So $L_5 \cap E_5$ is an example of an ω-language in $REG(X^\omega) \cap \mathbf{F}_\sigma \setminus (TB_f \cup DB_f)$. $\square$

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